

# TRANSFERS AND PERIODIC ORBITS OF HOMEOMORPHISMS

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**ABSTRACT.** Bo Ju Jiang applied Neilsen theory to the study of periodic orbits of a homeomorphism. His method employs a certain loop in the mapping torus of the homeomorphism. Our interest concerns the persistence of periodic orbits in parameterized families of homeomorphisms. This leads us to consider fibre bundles and equivariant maps, which gives us a nice point of view.

## 1. INTRODUCTION

Bo Ju Jiang introduced mapping tori into the study of periodic orbits of a homeomorphism in his article [Jiang(1996)]. If  $h : F \rightarrow F$  is a homeomorphism, then a periodic orbit of length  $n$  gives rise to a loop  $\sigma$  in the mapping torus  $T_h$  which represents an element  $[\sigma]$  in the fundamental group of the mapping torus  $\pi_1(T_h)$ . This element  $[\sigma]$  plays a role in the Neilsen theory for the mapping torus and gives information about the existence of a periodic orbit of the homeomorphism  $h$ .

Now the mapping torus  $T_h$  gives rise to a fibre bundle  $p : T_h \rightarrow S^1$  over a circle with fibre  $F$ . The loop  $\sigma$  can be regarded as an  $n$ -covering space over the base circle when the fibre bundle projection  $p$  is restricted to the image of  $\sigma$  in  $T_h$ .

We are interested in the persistence of periodic orbits under isotopies of  $h$ , or more generally, under parameterized families of homeomorphisms. This suggests that we consider fibre bundles with fibre  $F$  over a base  $B$  and ask when does the total space  $E$  contain a subspace  $S$  so the the restriction of the fibre bundle projection  $p : E \rightarrow B$  to  $S$  results in a map  $S \rightarrow B$  of “degree”  $n$ .

Note this generalizes the fact that  $\sigma$  maps to  $S^1$  with degree  $n$ . We also point out that when  $B$  is not an oriented manifold, we take the definition of degree of a map given in [Gottlieb(1986)]: The *degree* of a map  $f : X \rightarrow Y$  is taken to be the smallest positive integer  $N$  for which there exists a homomorphism  $\tau : H_*(Y) \rightarrow H_*(X)$  on integral homology so that the composition  $f_* \circ \tau$  is given by multiplication by  $N$ . If there is no such integer, than the degree is defined to be zero. The homomorphism  $\tau$  will be called a *transfer* for  $f$  for the purposes of this paper.

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In the case of a single fixed point, there is an integer invariant called the index of the fixed point which guarantees the persistence of a fixed point under homotopy if the index is nonzero. This is also true for zeros of vectorfields; see [Gottlieb and Samaranayake(1994)]. This persistence leads to transfers for fibre bundles with vectorfields tangent to the fibre, [Becker and Gottlieb(1991)], and to transfers for fibre bundles with fibre preserving maps, [Dold(1976)].

A more direct transfer related to the Lefschetz number and to Reidemeister torsion was studied by Wolfgang Lück in [Lück(1997)], but it involves algebraic  $K$ -theory groups instead of homology groups.

We therefore initiate a study of transfers in a setting which parameterizes Jiang's approach by recalling relevant facts about principal fibre bundles and their associated principal fibre bundles. Then we investigate a few simple consequences of the existence of transfers.

## 2. AN EXAMPLE

Now the circle group  $S^1$  acts on itself via a multiplication map  $\mu : S^1 \times S^1 \rightarrow S^1$  given by  $(e^{2\pi i a}, e^{2\pi i b}) \mapsto e^{2\pi i(a+b)}$ . Now left multiplication by  $e^{2\pi i a}$  is a self homeomorphism, and it has a periodic orbit of length equal to the denominator of  $a$  if  $a$  is a fraction with relatively prime numerator and denominator. If  $a$  is irrational, there is no periodic orbit.

On the other hand, the multiplication map  $\mu$  is the clutching map of a fibre bundle over the two dimensional sphere  $S^2$ . The fibre bundle is fibre bundle equivalent to the Hopf bundle

$$S^1 \rightarrow S^3 \rightarrow S^2$$

and since the second homology group of the total space is zero and the second homology group of the base space is infinite cyclic, the degree of the projection must be zero.

If we alter the map  $\mu$  by a homotopy, so that the the map of  $S^1$  into the space of self homeomorphisms of  $S^1$  corresponding to  $\mu$  is homotoped to another map  $S^1 \rightarrow \text{Homeo}(S^1)$ , the corresponding clutching map still is fibre bundle equivalent to the Hopf bundle, and so the degree is zero and so there is "no transfer". Hence we expect that there is no global "coherent" periodic orbit for the family of homeomorphisms that the clutching map represents. Later in this paper we will describe more precisely what we mean by coherent.

## 3. PRINCIPAL BUNDLES AND ASSOCIATED BUNDLES

A *principal bundle*  $G \rightarrow E \rightarrow B$  is a fibre bundle whose fibre is a group  $G$  and  $G$  acts freely on the total space  $E$ . Hence the quotient map  $q : E \rightarrow E/G$  is the fibre bundle projection.

Given an action of  $G$  on a space  $F$ , we consider the diagonal action of  $G$  on  $E \times F$ . Then the *Borel construction*  $E \times_G F$  is defined to be the quotient space of the diagonal action of  $G$  on  $E \times F$ . This results in the *associated fibre bundle*

$$F \rightarrow E \times_G F \rightarrow B$$

We can regard the particular action of  $G$  on  $F$  as a homomorphism

$$\rho : G \rightarrow \text{Homeo}(F)$$

from the group to the space of homeomorphisms of  $F$ . So a more precise notation for the Borel construction is  $E \times_\rho F$ .

**Lemma 1.** *A cross-section  $s$  to the associated bundle  $F \rightarrow E \times_G F \rightarrow B$  corresponds to a  $G$ -map  $\hat{s} : E \rightarrow F$*

*proof.* Let  $s : B \rightarrow E \times_\rho F$  be the cross-section. Then  $s(b) = [e, x] = [ge, gx]$  where the projection  $p : [e, x] \mapsto b$ . Then we define  $\hat{s} : E \rightarrow F$  by  $\hat{s} : e \mapsto x$ . This is indeed an equivariant map since  $[e, x] = [ge, gx]$  implies  $\hat{s}(ge) = gx$ .

Conversely, given an equivariant map  $\hat{s}$ , we define a cross-section by  $s : b \mapsto [e, \hat{s}(e)]$ . This is well defined since  $[e, \hat{s}(e)] = [ge, g\hat{s}(e)] = [ge, \hat{s}(ge)]$   $\square$

Important examples of this lemma are:

(1) If  $\rho : G \rightarrow \text{Homeo}(G)$  is given by right multiplication on  $G$  by itself, then the associated bundle  $G \rightarrow E \times_\rho G \rightarrow B$  is the principal bundle itself. Then a  $G$ -map  $E \rightarrow G$  gives a cross-section of the principal bundle. In this case, the principal bundle is trivial for two different reasons: The fibre is a retract of the total space; and the bundle has a cross-section.

(2) If  $\rho : G \rightarrow \text{Homeo}(G)$  is given by conjugation on  $G$ , then a cross-section to the associated bundle corresponds to a *principal bundle equivalence*  $h : E \rightarrow E$ . This is an equivariant map which induces the identity on the base space  $B$ . This follows because the cross-section corresponds to a  $G$ -map  $f : E \rightarrow G$ , where  $G$  acts on itself by the adjoint representation. Then the map  $h : e \mapsto f(e)e$  is the corresponding bundle equivalence since  $f(ge) = gf(e)g^{-1}$ .

Now Milnor(1956) showed that every topological group  $G$  has a *universal principal bundle*

$$G \rightarrow E_G \rightarrow B_G$$

(where  $E_G$  is contractible). This means that the set of principal  $G$ -bundle equivalence classes over a base space  $B$  are in one-to one correspondence with the homotopy classes  $[B, B_G]$  when  $B$  is a CW-complex. The correspondence is induced by assigning to a map  $f : B \rightarrow B_G$  the *pullback bundle*  $G \rightarrow f^*(E_G) \rightarrow B$  where the total space  $f^*E_G = \{(b, e) \in B \times E_G \mid f(b) = p(e)\}$ .

A related universal fibration holds for Hurewicz fibrations. If

$$F \rightarrow E_\infty \rightarrow B_\infty$$

is a universal Hurewicz fibration, then let  $E_\infty^{(F)}$  denote the space of maps of  $F \rightarrow E_\infty$  which are homotopy equivalences into fibres of  $E_\infty$ . Then  $E_\infty^{(F)} \rightarrow B_\infty$  is

a principal fibration whose fibre is the space of self-homotopy-equivalences. Now  $E_\infty^{(F)}$  is essentially contractible so  $B_\infty$  is the classifying space of the group of self-homotopy-equivalences of  $F$ . These results have topological issues attached to them. The most recent exposé is Booth(2000).

So we may consider the sequence of homomorphisms  $G \rightarrow \mathcal{H} \rightarrow \mathcal{E}$  where  $\mathcal{H}$  denotes the group of homeomorphisms of  $F$  and  $\mathcal{E}$  denotes the monoid of self homotopy equivalences of  $F$ . Then a fibre bundle with fibre  $F$  and group  $G$  may be discussed using the sequence of induced maps  $B \rightarrow B_G \rightarrow B_{\mathcal{H}} \rightarrow B_{\mathcal{E}}$ . That is, given a fibration over  $B$ , there is a classifying map  $k : B \rightarrow B_{\mathcal{E}}$ . This map factors, up to homotopy, through  $B_{\mathcal{E}}$  if and only if the fibration over  $B$  is fibre homotopy equivalent to a fibre bundle. Similarly, if  $k$  factors through  $B_G$ , then the fibre bundle is bundle equivalent to a bundle with structure group  $G$ .

Finally, we can classify the groups of self principal bundle equivalences and fibre homotopy equivalences. Let  $\mathcal{G}$  denote the topological group of self principal bundle equivalences. Then [Gottlieb(1972)] showed that for a principal or associated bundle over a base space  $B$  whose classifying map is  $k : B \rightarrow B_G$ , the classifying space of  $\mathcal{G}$  is the space of maps of the base space into the classifying space which are homotopic to  $k$ . In symbols:  $B_{\mathcal{G}} = \text{Map}(B, B_G; k)$ . The same result hold for self fibre homotopy equivalences, [Gottlieb(1968)] and [ Gottlieb(1970)]. See also [Booth, Heath, Morgan, Piccinini(1984)]. The group  $\mathcal{G}$  is also called the group of *gauge transformations* for the relevant principal bundle, [Atiyah, Bott(1983)].

#### 4. MAPPING TORI

The universal principal bundle for the integers  $\mathbb{Z}$  is the universal covering space of the circle,  $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$ . If  $h$  is a self homeomorphism of  $F$ , the homomorphism  $\rho$  from  $\mathbb{Z}$  to  $\mathcal{H}$  given by  $n \mapsto h^n$  gives rise to a map  $S^1 = B_{\mathbb{Z}} \rightarrow B_{\mathcal{H}}$ . This map pulls back to a principal  $\mathcal{H}$  bundle  $E$  over the circle. If we apply the Borel construction  $E \times_{\rho} F$  to this bundle, we get the mapping torus  $T_h$ . Equivalently, we may take the Borel construction  $E_{\mathcal{H}} \times_{\mathcal{H}} F$  to get the associated bundle with fibre  $F$  and then take the pullback by the classifying map to get the mapping torus.

Now if  $G$  acts on  $F$  and if a subspace  $A$  of  $F$  is invariant under the action, we obtain a sub-bundle  $E \times_G A$  contained in  $E \times_G F$ . In the case of the mapping torus  $T_h$ , if  $A$  is an orbit of  $h$ , it is invariant under the action of the representation of  $\mathbb{Z}$  on  $F$  and so we get a one dimensional sub-bundle in  $T_h$ . This is the origin of the loop  $\sigma$  in the mapping torus which plays an important role in [Jaing(1996)].

Jiang also considers the situation in which there is an isotopy from  $h : F \rightarrow F$  to the identity map. This leads to invariants such as braid groups. The isotopy corresponds to a bundle equivalence (or a gauge transformation using different words)  $f : T_h \rightarrow T_h$ . The fibre bundle  $F \rightarrow T_h \rightarrow S^1$  corresponds to a classifying map  $k : S^1 \rightarrow B_{\mathcal{H}}$ . The group of gauge transformations  $\mathcal{G}$  for this bundle has a classifying space  $B_{\mathcal{G}} = \text{Map}(S^1, B_{\mathcal{H}}; k)$ . The group of gauge transformations  $\mathcal{G}$  is homotopy equivalent to the loop space of its classifying space. Hence the path components of  $\mathcal{G}$  are in one-to-one correspondence to the fundamental group of the classifying space  $\pi_1(B_{\mathcal{G}}) = \pi_1(\text{Map}(S^1, B_{\mathcal{H}}; k))$ . In the case at hand, when  $h$  is isotopic to the identity, the map  $k$  must be homotopic to a constant map and the fibre bundle

must be trivial.

Now the unsettled part of Neilsen theory occurs in dimension 2. In the case that  $F$  is a closed connected oriented surface which is not the 2-sphere, M. E. Hamstrom has shown that the space of homeomorphisms has path components for which the higher homotopy groups are zero. That is each component is aspherical, [Hamstrom(1965), Hamstrom(1966)].

**Lemma 2.** *The space of maps  $\text{Map}(X, Y; k)$  where  $Y$  is a  $K(\pi, 1)$  is itself an aspherical CW complex with fundamental group  $\pi_1(\text{Map}(X, Y; k))$  equal to the centralizer of the image of  $k_* : \pi_1(X) \rightarrow \pi_1(Y) = \pi$ .*

*proof.* In [Gottlieb(1965)], it is shown that the identity component  $\text{Map}(Y, Y; 1_Y)$  is aspherical when  $Y$  is aspherical, and the fundamental group is isomorphic to the center of the fundamental group of  $Y$ . The argument that the mapping space is aspherical does not depend on the domain CW complex. The argument that the fundamental group is the center only depends on the fact that for any  $\alpha$  in the fundamental group of the domain  $X$ , its image  $k_*(\alpha)$  must commute with all of the fundamental group of  $Y$ . This is Lemma 2 of [Gottlieb(1969)].

Now suppose  $F$  is a  $K(\pi, 1)$ . Then lemma 2 implies that any component of the space of self homotopy equivalences is a  $K(Z\pi, 1)$  where  $Z\pi$  is the center of  $\pi$ . Hence  $B_{\mathcal{E}}$  has fundamental group equal to the group of outer automorphisms  $\text{Out}(\pi)$ , and the second homotopy group equal to the center of  $\pi$ . Thus if  $\pi$  has trivial center,  $B_{\mathcal{E}}$  is also aspherical. Thus the group of self fibre homotopy equivalence classes of the mapping torus of  $h$  for  $F$  an aspherical complex, denoted  $\pi_0(\mathcal{G})$ , is isomorphic to  $\pi_1(\text{Map}(S^1, B_{\mathcal{E}}; k))$  which in turn is isomorphic to the centralizer of the image of  $k_* : \pi_1(S^1) \cong \mathbb{Z} \rightarrow \pi_1(B_{\mathcal{E}}) \cong \text{Out}(\pi_1(F))$ .

On the mapping level, this shows that the fibre homotopy classes of fibre homotopy equivalences of  $T_h$  correspond to those homotopy equivalences  $f : F \rightarrow F$  so that  $f \circ h \sim h \circ f$  where  $\sim$  denotes “is homotopic to”.

Similarly, for  $F$  a closed connected orientable surface with negative Euler-Poincare number,  $B_{\mathcal{H}}$  is an aspherical space by [Hamstrom(1966)] and the fact that the fundamental group of  $F$  has trivial center. Hence the fibre isotopy classes of gauge transformations of  $T_h$  form a group isomorphic to the group of isotopy classes of self homeomorphisms of  $F$  which commute with  $h$ .

Where as an orbit of  $h$  in  $F$  gives rise to a subbundle of  $T_h$ , a gauge transformation does not preserve that subbundle in general, but embeds it into a homeomorphic subbundle. This situation must give rise to embedding type invariants, such as braid groups.

## 5. BUNDLES OVER SPHERES

The obvious generalization of mapping tori are bundles over spheres. Here the homeomorphism  $h : F \rightarrow F$  is replaced by the clutching map  $c : F \times S^{n-1} \rightarrow F$  (which corresponds to  $\hat{c} : S^{n-1} \rightarrow \mathcal{H}$ ), and the unit interval is replaced by the unit  $n$ -ball  $D^n$ . Then identifying  $F \times D^n$  to  $E'$  by using the clutching map on  $F \times \partial D^n$ , we obtain a fiber bundle  $F \rightarrow E' \rightarrow S^n$ . Now this fibre bundle has an associated

principal bundle  $\mathcal{H} \rightarrow E \rightarrow S^n$ . So the associated bundle  $E \times_{\mathcal{H}} F$  is fibre bundle equivalent to  $E'$ .

Now if  $\hat{c}$  maps into a subgroup  $G$  of  $\mathcal{H}$  which preserves a subspace  $A \subset F$ , then  $E \times_G A$  is a subbundle of  $E'$ . If  $A$  is a point, then the fibre of this subbundle is a point, and so there is a cross-section to  $E' \rightarrow S^n$ . Thus by lemma 2, there is a  $G$ -map from  $E$  to  $F$ .

Not all cross-sections of  $E'$  correspond to a  $G$ -subbundle. But every one corresponds to a  $G$ -map  $E \rightarrow F$ .

Now if  $E'$  contains an oriented closed submanifold  $M$  of dimension  $n$  which maps onto  $S^n$  under the projection with degree  $m$ , then the pullback  $p^* : E' \rightarrow S^n$  over  $M$  has a cross-section. So  $p^*E'$  has a  $G$  map into  $F$ .

These submanifolds give rise to transfers of  $E' \rightarrow S^n$  of trace  $m$ . In case we have a fibre preserving map  $f : E' \rightarrow E'$  inducing the identity on the base with Lefschetz number  $\Lambda_f$ , there will be such a manifold, [Gottlieb(1977)].

A corollary of this is: For every fibre bundle over a base space which is a closed oriented manifold (such as  $S^n$ ) and whose fibre is a closed manifold whose fibre has Euler-Poincare number equal to some nonzero integer  $m$ , then there is a manifold which maps onto the base space by a map of degree  $m$  so that the pullback bundle has a cross-section. Hence the associated principal bundle's total space maps equivariantly into the fibre  $F$ .

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